QUADRATIC DIRICHLET CHARACTERS

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Let q be odd prime , and let $m \in \mathbb{F}_q[T]$ be a polynomial of positive degree. We define

$$\chi_m(f) := \left(\frac{m}{f}\right)_2$$

Proposition 1. *Assume that m is monic, and $\deg m$ is even. Then χ_m is a Dirichlet character modulo m.*

Proof. Mutiplicativity is clear from properties of the Jacobi symbol. What needs checking is that $\chi_m(f+mg) = \chi_m(f)$.

We use quadratic reciprocity (for the Jacobi symbol): If $\gcd(f,m) = 1$, then

$$\chi_m(f) = \left(\frac{m}{f}\right)_2 = \left(\frac{f}{m}\right)_2 (-1)^{\frac{q-1}{2}\deg m \deg f} \operatorname{sign}_2(f)^{\deg m} \operatorname{sign}_2(m)^{-\deg f}$$

where

$$\operatorname{sign}_2(f) = (\text{Leading coefficient of } f)^{(q-1)/2}$$

For monic m, we have $\operatorname{sign}_2(m) = 1$. If $\deg m$ is even, then $\operatorname{sign}_2(f)^{\deg m} = (f_{\text{leading}})^{\frac{q-1}{2}\deg m} = 1$, and $(-1)^{\frac{q-1}{2}\deg m \deg f} = 1$. Thus in our case

$$\chi_m(f) = \left(\frac{f}{m}\right)_2 \tag{1}$$

which depends only on $f \bmod m$. □

Thus we have found several examples of real characters.

Definition 2. *A Dirichlet character χ is "even" if $\chi(cf) = \chi(f)$ for all scalars $c \in \mathbb{F}_q^\times$. Equivalently, if $\chi(c) = 1$ $\forall c \in \mathbb{F}_q^\times$.*

We claim that χ_m is an "even" character, that is $\chi_m(cf) = \chi_m(f)$, for $c \in \mathbb{F}_q^\times$. Indeed, using (1), $\chi_m(c) = \left(\frac{c}{m}\right)_2$ so that

$$\chi_m(c) = \left(\frac{c}{m}\right)_2 = c^{(|m|-1)/2}$$

by general properties of the Legendre symbol. But if $\deg m = 2\mu$ is even, then

$$\frac{|m|-1}{2} = \frac{q^{\deg m}-1}{2} = (q^\mu - 1)\frac{q^\mu+1}{2} = (q-1)(1+q+\cdots+q^{\mu-1}))\frac{q^\mu+1}{2}$$

is an integer multiple of $q-1$, and hence $c^{(|m|-1)/2} = (c^{q-1})^* = 1$ in $\mathbb{F}_q$. Thus $\chi_m(c) = 1$.

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Proposition 3. *For even Dirichlet characters* $\chi \bmod Q$, $L(0,\chi) = 0$. *In particular we can write*

$$L(s,\chi) = (1-q^{-s})P(q^{-s})$$

where $P(u)$ *is a polynomial of degree* $\leq \deg Q - 2$, *with* $P(0) = 1$.

Proof. We recall that for any nontrivial character $\chi \bmod Q$, we saw ("the analytic continuation") that

$$L(s,\chi) = \sum_{0 \leq n \leq \deg Q - 1} \Big(\sum_{\substack{\deg f = n \\ \text{monic}}} \chi(f) \Big) q^{-ns}$$

and therefore

$$L(0,\chi) = \sum_{\substack{\deg f \leq \deg Q - 1 \\ \text{monic}}} \chi(f)$$

Since χ is even, we have for all f,

$$\chi(f) = \frac{1}{q-1} \sum_{c \in \mathbb{F}_q^\times} \chi(cf)$$

and as f goes over all monic polynomials of given degree, the set $\{cf : c \in \mathbb{F}_q^\times\}$ will go over all polynomials of that degree (not necessarily monic). Thus

$$L(0,\chi) = \frac{1}{q-1} \sum_{\deg f \leq \deg Q - 1} \chi(f)$$

the sum now over all polynomials of degree strictly less than $\deg Q$. But this set is precisely all representatives of the residue classes modulo Q. This

$$L(0,\chi) = \frac{1}{q-1} \sum_{f \bmod Q} \chi(f)$$

By the orthogonality relations, this sum vanishes if $\chi \neq \chi_0$ is a nontrivial character.

Now $L(s,\chi) = \mathrm{Pol}(q^{-s})$ is a polynomial in $u = q^{-s}$, with $\mathrm{Pol}(u) = 1 + *u + \cdots + *u^{\deg Q - 1}$. Since $\mathrm{Pol}(1) = 0$ (evaluate at $q^{-0} = 1$) we have by general properties of polynomials that $\mathrm{Pol}(u) = (1-u)P(u)$ for some other polynomial, of degree one less. And since $\mathrm{Pol}(0) = 1$ we also need $P(0) = 1$. □

Corollary 4. *If* $\deg m = 2$, m *monic then*

$$L(s,\chi_m) = 1 - q^{-s}$$

Proof. Since χ_m is an even Dirichlet character, and $\deg m = 2$, we use Proposition 3 to deduce that $L(s,\chi_m) = (1-q^{-s})P(q^{-s})$ and now $P(u)$ has degree $\deg m - 2 = 0$ so is a scalar, and $P(0) = 1$ so $P(u) \equiv 1$. □

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